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Note

A decidability result for the dominating set problem

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ABSTRACT

We study the following question: given a finite collection of graphs $G_1, \ldots, G_k$, is the dominating set problem polynomial-time solvable in the class of $(G_1, \ldots, G_k)$-free graphs? In this paper, we prove the existence of an efficient algorithm that answers this question for $k = 2$.

1. Introduction

A set of vertices in a graph is *dominating* if every vertex outside the set has a neighbor in the set. The dominating set problem is that of finding in a graph a dominating set of minimum cardinality. This is a classical algorithmic problem with numerous applications (see e.g. [10]). From a computational point of view, this problem is difficult, i.e. it is NP-hard. Moreover, it remains NP-hard under substantial restrictions, for instance, for bounded degree graphs, bipartite graphs, split graphs [3], line graphs [18], etc. On the other hand, for graphs in some special families, such as P_4-free graphs, (K_p, P_5)-free graphs [19], convex bipartite graphs [4] or the complements of bipartite graphs, the problem admits polynomial-time solutions.

Paper [13] provides a complete classification of graph families defined by a single forbidden induced subgraph with respect to the computational complexity of the problem. However, no such classification is known for families defined by more than one forbidden induced subgraph. In the present paper, we analyze this question from a decidability point of view: given a finite collection of graphs $G_1, \ldots, G_k$, is the dominating set problem polynomial-time solvable in the class of $(G_1, \ldots, G_k)$-free graphs? We employ the notions of limit classes and well-quasi-orders to prove the existence of an efficient algorithm answering the above question in the case of two forbidden induced subgraphs. Our proof is not constructive and does not provide any specific procedure for solving the problem. It only proves the existence of a solution in the spirit of recognizability of minor-closed graph classes derived from the finiteness of sets of minimal excluded minors [17] (see also [7,8] for more results on non-constructive tools for proving polynomial-time decidability).

The main result of the paper is proved in Section 3, while Section 2 provides preliminary information related to the topic. Section 4 concludes the paper with a few open problems.

2. Preliminary results

All graphs in this paper are undirected, without loops or multiple edges. In a graph, an independent set is a subset of pairwise non-adjacent vertices and a clique is a subset of pairwise adjacent vertices. As usual, P_n, C_n and K_n denote a chordless

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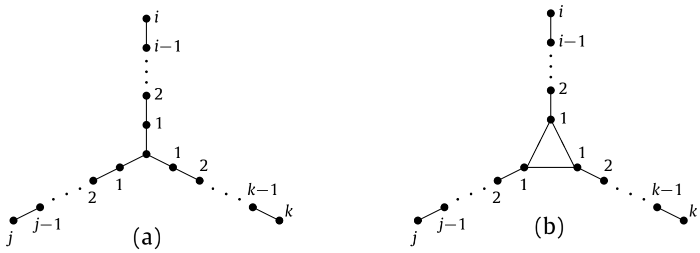

Fig. 1. The graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b).

path, a chordless cycle and a complete graph on n vertices, respectively. Given two graphs G and H, we denote by $G+H$ the disjoint union of G and H. In particular, mG is the disjoint union of m copies of G.

A graph G is an induced subgraph of a graph H if G can be obtained from H by vertex deletion. A class of graphs is *hereditary* if with any graph G it contains all induced subgraphs of G. Examples of hereditary classes include all classes mentioned in the introduction (graphs of degree at most d, bipartite graphs, split graphs, line graphs, P_4-free graphs) and many other important classes such as planar, perfect, interval graphs, all monotone classes (i.e. classes closed under vertex and edge deletion), all minor-closed graph classes (i.e. classes closed under vertex deletion, edge deletion, and edge contraction), etc.

It is known that a class of graphs is hereditary if and only if it can be characterized by a set of forbidden induced subgraphs, i.e. minimal graphs that are not in the class. The class of graphs containing no induced subgraphs from a set X will be denoted as $Free(X)$, and graphs in the class $Free(X)$ will be called X-free. If X is a finite set, we say that $Free(X)$ is a *finitely defined* class of graphs. Among the examples mentioned above, graphs of degree at most d, split graphs, line graphs, and P_4-free graphs are finitely defined. For instance, the class of line graphs is characterized by nine forbidden induced subgraphs [15], and the class of split graphs (i.e. graphs whose vertices can be partitioned into an independent set and a clique) is precisely $Free(2K_2, C_4, C_5)$ [9]. If X consists of a single graph, we call $Free(X)$ a *monogenic* class of graphs. The P_4-free graphs (also known as cographs) provide an example of a monogenic class.

As we mentioned in the introduction, in some hereditary classes of graphs the dominating set problem is NP-hard and in some others it can be solved in polynomial time. Under the assumption that $P \neq NP$, our goal is to develop a tool that would be helpful in deciding whether a given class of graphs is "simple" or "difficult" for the problem in question. To this end, we employ the notion of limit classes of graphs defined as follows.

Definition 1.

- We call a class X of graphs *DS-tough* if there is no polynomial-time algorithm to solve the dominating set problem for graphs in X.
- The intersection of any sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ of (not necessarily distinct) DS-tough classes of graphs is called a limit class for the dominating set problem.

The importance of the notion of a limit class is due to the following result.

Theorem 1. *A finitely defined class X of graphs is DS-tough if and only if X contains a limit class for the dominating set problem.*

Proof. One direction of the theorem is trivial, since any DS-tough class is a limit class by definition. Assume now that a finitely defined class $Y = Free(G_1, \ldots, G_k)$ contains a limit class X. Let X be the intersection of a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ of DS-tough classes. Since Y contains X, for each graph G_i which is forbidden for Y there must exist a graph H_i which is forbidden for X and which is an induced subgraph of G_i. Moreover, there must exist a number ℓ_i such that X_{ℓ_i} does not contain H_i, since otherwise H_i belongs to all classes of the sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ and therefore to X. Defining $\ell = \max\{\ell_1, \ldots, \ell_k\}$, we conclude that X_ℓ does not contain the graphs $G_1, \ldots, G_k$, i.e. Y contains X_ℓ, which means that Y is DS-tough. $\square$

Two limit class for the dominating set problem have been identified in [13]. These are:

$\mathcal{S}$ the class of graphs every connected component of which is of the form $S_{i,j,k}$ (Fig. 1 (a));
$\mathcal{T}$ the class of graphs every connected component of which is of the form $T_{i,j,k}$ (Fig. 1 (b)).

One more limit class was found in [1]. To define this class, let us emphasize that $\mathcal{S}$ is a class of bipartite graphs. Assume that every graph G in $\mathcal{S}$ is given together with a bipartition of its vertex set into two independent sets A and B such that all vertices of degree 3 belong to A. By connecting every two vertices of A by an edge we create a split graph denoted as G^*. Then the third limit class for the dominating set problem is:

$\mathcal{Q}$ the class of split graphs G^* such that $G \in \mathcal{S}$.

Moreover, in [1] it was shown that $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$ are minimal limit classes, i.e. none of them contains a smaller limit classes. In Section 4, we show that there are more minimal limit classes for the dominating set problem. However, in the case of monogenic classes of graphs, Theorem 1 can be restricted, without loss of generality, to the three particular classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$. More specifically:

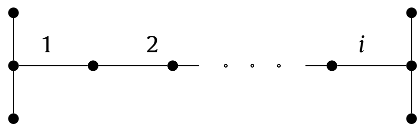

Fig. 2. Graph H_i.

Theorem 2. *A monogenic class of graphs is DS-tough if and only if it contains one of the three classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$.*

Proof. Let G be a graph. If $Free(G)$ contains one of the classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$, then $Free(G)$ is DS-tough by Theorem 1.

Now assume $Free(G)$ contains none of the classes $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$, i.e. G belongs to each of these three classes. Then it is not difficult to see that $G = P_k + tK_1$ with $k \leq 4$ and $t \geq 0$. If $t = 0$, then $Free(G)$ is a (sub)class of cographs, in which case the problem is known to be polynomial-time solvable (see e.g. [6]). The more general case of $t > 0$ was shown to be solvable in [13]. $\square$

In the next section, we show that for graph classes defined by two forbidden induced subgraphs the situation is somewhat similar to the monogenic case. To be more precise, let us introduce the following definition.

Definition 2. A family Φ of limit classes will be called k-tight if every DS-tough class defined by at most k forbidden induced subgraphs contains a limit class from the set Φ.

Theorem 2 proves that $\{\mathcal{S}, \mathcal{T}, \mathcal{Q}\}$ is a 1-tight family. In the next section, we will show that there exists a finite 2-tight family. A crucial role in the proof of this result is played by the notion of a well-quasi-order.

A binary relation $\leq$ on a set X is a *quasi-order* if it is reflexive and transitive. Two elements $x, y \in X$ are said to be incomparable if neither $x \leq y$ nor $y \leq x$ holds. An *antichain* in a quasi-order is a set of pairwise incomparable elements. A quasi-order $(X, \leq)$ is a *well-quasi-order* if X contains no infinite strictly decreasing sequences and no infinite antichains.

In this paper, we study the induced subgraph relation, which is not a well-quasi-order on the set of all graphs. Indeed, the cycles $C_3, C_4, C_5 \ldots$ and the graphs $H_1, H_2, H_3 \ldots$ (see Fig. 2) form infinite antichains under the induced subgraph relation. However, restricted to some special classes, the induced subgraph relation becomes a well-quasi-order. Of particular interest to us is the following conclusion.

Theorem 3. *The classes $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{Q}$ are well-quasi-ordered by the induced subgraph relation.*

Proof. Observe that the class $\mathcal{S}$ is not only hereditary but also monotone (i.e. closed under deletion of vertices and edges). In [5], it was shown that a monotone class of graphs is well-quasi-ordered by the induced subgraph relation if and only if it contains finitely many cycles and graphs of the form H_i. The class $\mathcal{S}$ contains no such graphs and therefore it is well-quasi-ordered by the induced subgraph relation.

The class $\mathcal{T}$ contains no cycles of length more than 3 and no graphs of the form H_i. However, this class is not monotone. Extending it to a monotone class (by adding to it all subgraphs, not necessarily induced, of graphs in $\mathcal{T}$) does not add any new cycles or graphs of the form H_i. Therefore, $\mathcal{T}$ is also well-quasi-ordered by induced subgraphs.

To see that $\mathcal{Q}$ is well-quasi-ordered, observe that a graph $G \in \mathcal{S}$ is an induced subgraph of the graph $H \in \mathcal{S}$ if and only if $G^* \in \mathcal{Q}$ is an induced subgraph of $H^* \in \mathcal{Q}$. $\square$

In the next section, we study classes defined by two forbidden induced subgraphs. We consider the set of ordered pairs of graphs as being quasi-ordered under the product ordering: $(G_1, H_1) \preceq (G_2, H_2)$ if and only if G_1 is an induced subgraph of G_2 and H_1 is an induced subgraph of H_2.

From the theory of well-quasi-orders, we will need the following fact, which follows from Higman's lemma [11].

Fact 1. *The product of finitely many well-quasi-orders is a well-quasi-order.*

3. Main result

Lemma 1. *For the dominating set problem, there is a finite 2-tight family of limit classes.*

Proof. For every DS-tough class $Free(G, H)$, fix a limit class $B_{G,H}$ contained in $Free(G, H)$. For each pair X, Y of classes from the triple $\{\mathcal{S}, \mathcal{T}, \mathcal{Q}\}$, we give the following definitions:

- $L(X, Y) = \{(G, H) \ : \ G \in X, H \in Y$ and $Free(G, H)$ is DS-tough$\}$,
- $L^*(X, Y) =$ the set of minimal pairs from $L(X, Y)$,
- $\mathcal{B}(X, Y) = \{B_{G,H} \ : \ (G, H) \in L^*(X, Y)\}$.

Since each of the classes $\mathcal{S}, \mathcal{T}, \mathcal{Q}$ is well-quasi-ordered, the set $L^*(X, Y)$, like the set $\mathcal{B}(X, Y)$, is finite. Therefore, the set

$$\mathcal{B} = \{\mathcal{S}, \mathcal{T}, \mathcal{Q}\} \cup \mathcal{B}(\mathcal{S}, \mathcal{T}) \cup \mathcal{B}(\mathcal{S}, \mathcal{Q}) \cup \mathcal{B}(\mathcal{T}, \mathcal{Q})$$

is also finite. To prove the theorem, we will show that for any two graphs G and H, the class $Free(G, H)$ is DS-tough if and only if it contains a limit class from the set $\mathcal{B}$. One direction follows readily from Theorem 1, i.e. if $Free(G, H)$ contains a limit class from $\mathcal{B}$, then $Free(G, H)$ is DS-tough.

Now assume that $Free(G, H)$ is DS-tough. Then the classes $Free(G)$ and $Free(H)$ are also DS-tough. Therefore, by Theorem 2, there is a class $X \in \{\mathcal{S}, \mathcal{T}, \mathcal{Q}\}$ contained in $Free(G)$ and a class $Y \in \{\mathcal{S}, \mathcal{T}, \mathcal{Q}\}$ contained in $Free(H)$. If additionally X is contained in $Free(H)$ or Y is contained in $Free(G)$, then $Free(G, H)$ contains X or Y, respectively. If neither X is contained in $Free(H)$ nor Y is contained in $Free(G)$ holds, then $H \in X$ and $G \in Y$, and therefore $Free(G, H)$ contains a class from $\mathcal{B}(X, Y)$. □

Theorem 4. *There is a polynomial-time algorithm that, given two graphs G and H, decides whether the class Free(G, H) is DS-tough or not.*

Proof. From the proof of Theorem 1 we know that the class of (G, H)-free graphs is DS-tough if and only if it contains one of the limit classes in the set

$$\mathcal{B} = \{\mathcal{S}, \mathcal{T}, \mathcal{Q}\} \cup \mathcal{B}(\mathcal{S}, \mathcal{T}) \cup \mathcal{B}(\mathcal{S}, \mathcal{Q}) \cup \mathcal{B}(\mathcal{T}, \mathcal{Q}).$$

Checking whether $Free(G, H)$ contains a class from the set $\{\mathcal{S}, \mathcal{T}, \mathcal{Q}\}$ is a polynomially solvable task, since graphs in each of the classes $\mathcal{S}, \mathcal{T}, \mathcal{Q}$ can be recognized in polynomial time, which is a trivial observation.

Checking whether $Free(G, H)$ contains a class from the set $\mathcal{B}(\mathcal{S}, \mathcal{T}) \cup \mathcal{B}(\mathcal{S}, \mathcal{Q}) \cup \mathcal{B}(\mathcal{T}, \mathcal{Q})$ is equivalent to checking whether there is a pair of graphs $(G', H') \in L^*(\mathcal{S}, \mathcal{T}) \cup L^*(\mathcal{S}, \mathcal{Q}) \cup L^*(\mathcal{T}, \mathcal{Q})$, such that $(G', H') \preceq (G, H)$. Since the set $L^*(\mathcal{S}, \mathcal{T}) \cup L^*(\mathcal{S}, \mathcal{Q}) \cup L^*(\mathcal{T}, \mathcal{Q})$ is finite, this can obviously be done in time polynomial in the size of the input graphs G and H. □

4. Concluding remarks and open problems

In this paper we proved the existence of an efficient algorithm for answering the following question: given two graphs G and H, is the dominating set problem polynomial-time solvable in the class of (G, H)-free graphs? Answering this question for classes defined by more than two forbidden induced subgraphs is a natural open problem. To solve it, we need to know more about limit classes. Let us show that for the dominating set problem there exists at least one more minimal limit class. It is known that the problem is NP-hard in the class of chordal bipartite graphs, i.e. in the class $Free(C_3, C_5, C_6, C_7 \dots)$ [16]. Therefore, there must exist a minimal limit class $\mathcal{X}$ contained in chordal bipartite graphs and a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ of DS-tough subclasses of chordal bipartite graphs converging to $\mathcal{X}$. Obviously, the class $\mathcal{S}$ is a subclass of chordal bipartite graphs, but $\mathcal{X}$ must be different from $\mathcal{S}$. Indeed, each class in the sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ must contain a C_4, since otherwise it is not DS-tough (being a subclass of forests), but then the class $\mathcal{X}$ contains a C_4, which is not the case for the class $\mathcal{S}$.

It is also natural to ask whether similar results can be obtained with respect to other algorithmic graph problems. For instance, paper [14] provides a complete classification of monogenic classes of graphs with respect to the complexity of the vertex coloring problem. Some limit classes for this problem have been identified in [2,12]. It is not difficult to show that the family of limit classes identified in these papers is 1-tight. However, not all of them are well-quasi-ordered by the induced subgraph relation. Establishing whether there is a 1-tight family of well-quasi-ordered limit classes for this problem is a challenging research problem.

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